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LETTER TO THE EDITOR

Finite-size effects for nested Bethe ansatz equations: analytical and numerical results for SU(N) and O($2N$) magnets

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Abstract. We study the finite-size effects in the SU(N) and O($2N$) Bethe ansatz equations for ground-state configurations. We compare our results with a numerical solution of the associated Bethe ansatz equations.

Recently there has been a strong growth of interest in the study of Bethe ansatz equations (BAE) for a finite-size lattice (L). There is a large class of integrable gapless models soluble by the Bethe ansatz approach, and in this case the computation of finite-size effects for the eigenspectrum makes possible the calculation of conformal properties of these systems [1]. Another interest in this subject is the analysis of the deviations from the famous 'string' picture [2], usually assumed for calculating some properties of these integrable models in the thermodynamic limit. De Vega and Woynarovich [3] proposed a systematic method to compute the finite-size effects for the eigenspectrum of the integrable models when the solution of the associated BAE is characterised by a set of real roots. This method was generalised to include the nested Bethe ansatz equations [4] and for the spin-$\frac{1}{2}$ XXZ model with different types of boundary conditions [5]. More recently De Vega and Woynarovich [6] studied the finite-size effects in the BAE for the SU(2) spin-s Heisenberg model, reformulating their previous method [3] in order to include the case where the set of BAE roots is complex. In this letter we use this generalised method to study the effects of finite size in the imaginary part of the nested BAE roots for SU(N) and O($2N$) integrable magnets.

The nested BAE for SU(N) integrable models is given by [7]:

$$\prod_{\substack{p=1\\ j\neq p}}^{n_r} \frac{\lambda_j^r-\lambda_p^r-\mathrm{i}}{\lambda_j^r-\lambda_p^r+\mathrm{i}} \prod_{l\in L_r} \prod_{p=1}^{n_l} \frac{\lambda_j^r-\lambda_p^l+\mathrm{i}[(m-1)\delta_{r,1}+1]/2}{\lambda_j^r-\lambda_p^l-\mathrm{i}[(m-1)\delta_{r,1}+1]/2}=1 \tag{1}$$

where $j=1,\ldots,n_r$, $r=1,\ldots,N-1$, $L_r=\{r-1,r+1\}$, $\lambda_j^0=n_N^0=0$ and $n_0=L$ is the lattice size.

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The ground state of this system (1), for finite L, is characterised by a set of roots $\lambda_{j,p}^r$, given by

$$\lambda_{j,p}^r = \lambda_j^r + \tfrac{1}{2}\mathrm{i}(m-2p-1) + \mathrm{i}\delta_{j,p}^r \tag{2}$$

with $p=0,\ldots,m-1$, $r=1,\ldots,N-1$ and $\delta_{j,p}^r = -\delta_{j,m-p-1}^r$. In (2) the λ_j^r are real numbers and $\delta_{j,p}^r$ are the deviations from the solutions in the thermodynamic limit $L\to\infty$ (string picture). Strictly at $L\to\infty$ (1) and (2) can be manipulated for a given set of $N-1$ coupled integral equations for the densities $\sigma^r(\lambda_j^r)$ of λ_j^r, $r=1,\ldots,N-1$. In this case these integral equations can be solved by standard Fourier techniques, and the $\sigma^r(x)$ are given by [7]:

$$\sigma^r(x) = \frac{1}{N}\frac{\sin[\pi(N-r)/N]}{\cosh(2\pi x/N)+\cos[\pi(N-r)/N]}. \tag{3}$$

For finite L the first non-null $\delta_{j,p}^r$ deviations appear at $m=2$. In this case the roots of (2) can be rewritten as

$$\lambda_j^{r,\pm} = \lambda_j^r \pm \mathrm{i}(\tfrac{1}{2}+\delta_j^r) \qquad r=1,\ldots,N-1 \tag{4}$$

where $+$, $-$ means $k=1,2$ in (2), respectively. The δ_j^r dependence of finite L can be calculated using the procedure developed in [6]. First we substitute (2) into (1), and taking the logarithm we may transform the products (1) into sums. The evaluation of these sums can be done using an extended Euler–Maclaurin formula that includes important non-analytical effects in $O(1/L)$ [6], and here we give only the final results. The deviations δ_j^r satisfy a set of coupled equations, given by

$$\begin{aligned}\pi\alpha_0^r(\lambda_j^r) &+ \tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x\, \alpha_0^r(x)[\psi_1'(\lambda_j^r - x) + \phi_1'(\lambda_j^r - x)] \\ &- \tfrac{1}{2}\sum_{p=r+1,r-1}\int_{-\infty}^{+\infty} \mathrm{d}x\, \alpha_0^p(x)[\psi_{1/2}'(\lambda_j^r - x) + \phi_{1/2}'(\lambda_j^r - x)] \\ &+ \log[1-\mathrm{e}^{-2\pi\alpha_0^r(\lambda_j^r)}] = 0\end{aligned} \tag{5}$$

where $\alpha_p^r(x) = 2L\sigma^r(x)\delta_p^r(x)$, $r=1,\ldots,N-1$ and $\alpha_p^N(x)=\alpha_p^0(x)=0$. The functions $\psi_a(x)$ and $\phi_a(x)$ differ only in the cut structure [6], and are defined by

$$\psi_a(x) = \frac{1}{\mathrm{i}}\log\left(\frac{1+\mathrm{i}x/a}{1-\mathrm{i}x/a}\right) \qquad \phi_a(x) = \frac{1}{\mathrm{i}}\log\left(\frac{x-\mathrm{i}a}{x+\mathrm{i}a}\right). \tag{6}$$

Equation (5) admits the choice $\alpha_0^r(x)=\alpha_0^r$, independent of variable x. In this case the integration upon x can be easily done, and we have:

$$2\pi\alpha_0^r - \pi(\alpha_0^{r+1}+\alpha_0^{r-1}) = -\log[1-\mathrm{e}^{-2\pi\alpha_0^r}]. \tag{7}$$

The solution of these equations (7) for α_0^1 and arbitrary N is

$$\alpha_0^1 = \frac{1}{\pi}\ln\left\{\left[\cos\left(\frac{(\pi/2)[(N/2)-1]}{(N/2)+1}\right)\right]\left[\cos\left(\frac{N\pi/4}{(N/2)+1}\right)\right]^{-1}\right\} \tag{8}$$

and the other α_0^r, $r=2,\ldots,N-1$ can be determined using (8) in (7). In order to verify these analytical calculations we solve numerically (1) [8]. In table 1(a) we compare the analytical results (8) with the numerical solution of (1) for δ_j^r in the case $N=3,4$. The analytical results are better for roots where the real part is not too close to the

Table 1. The deviations for numerical (δ_0^r) and analytical (Δ_0^r) calculations for SU(N) group with $N=3,4$ and (a) $m=2$, (b) $m=3$. Here we consider the lattice size $L=24$ and the deviations are in the crescent order in index r ($\delta_0^1, \delta_0^2, \ldots, \delta_0^{N-1}$).

(a)	$m=2$			(b)	$m=3$		
SU(3)		SU(4)		SU(3)		SU(4)	
δ_0^r	Δ_0^r	δ_0^r	Δ_0^r	δ_0^r	Δ_0^r	δ_0^r	Δ_0^r
0.069 83	0.076 81	0.099 02	0.104 47	0.101 26	0.117 53	0.146 50	0.161 67
0.022 79	0.024 43	0.031 87	0.033 35	0.032 93	0.036 48	0.047 22	0.050 89
0.013 26	0.013 78	0.017 92	0.018 54	0.019 09	0.020 31	0.026 48	0.027 88
0.009 45	0.009 71	0.012 32	0.012 64	0.013 58	0.014 25	0.018 18	0.018 93
0.007 51	0.007 68	0.009 46	0.009 67	0.010 79	0.011 24	0.013 96	0.014 44
0.006 44	0.006 56	0.007 83	0.007 98	0.009 24	0.009 58	0.015 50	0.011 91
0.005 84	0.005 94	0.006 87	0.006 99	0.008 39	0.008 68	0.010 12	0.010 42
0.005 58	0.005 67	0.006 32	0.006 42	0.008 01	0.008 28	0.009 31	0.009 57
0.076 31	0.083 00	0.006 07	0.006 17	0.110 76	0.126 81	0.008 94	0.009 19
0.028 58	0.030 09	0.116 91	0.140 06	0.041 19	0.044 64	0.174 06	0.176 66
0.019 67	0.020 21	0.048 56	0.053 92	0.028 25	0.029 66	0.072 67	0.083 77
0.016 93	0.017 28	0.030 28	0.031 60	0.024 31	0.025 27	0.045 14	0.048 12
		0.023 21	0.023 94			0.034 54	0.036 22
		0.019 91	0.020 39			0.029 61	0.030 76
		0.018 55	0.018 95			0.027 57	0.028 55
		0.115 94	0.119 97			0.172 06	0.185 30
		0.047 64	0.049 16			0.070 39	0.074 09
		0.036 24	0.037 00			0.053 43	0.055 34

ends of the distributions of λ_j^r, in agreement with [6]. It is possible to generalise these calculations for arbitrary m, and now we have

$$\begin{aligned} &2\pi\alpha_p^r - \pi(\alpha_p^{r+1}+\alpha_p^{r-1}) = -\log f_p^r(\alpha_p^r, \alpha_{p+1}^r, \alpha_{p-1}^r) \\ &f_p^r = \frac{1-\exp[-\pi(\alpha_p^r-\alpha_{p+1}^r)]}{1-\exp[-\pi(\alpha_{p-1}^r-\alpha_p^r)]} \qquad p=1,\ldots[m/2]-1 \\ &f_0^r = 1-\exp[\pi(\alpha_0^r-\alpha_1^r)] \end{aligned} \tag{9}$$

where $[m/2]$ is the integer part of the ratio $m/2$, and $\alpha_p^r = -\alpha_{m-1-p}^r$. For $m=3$ we have $\alpha_0^r = -\alpha_2^r$, $\alpha_1^r = 0$, and in table 2 we show some values of constants α_0^r for $N=3$ and $N=4$. In table 1(b) we compare the results (9) and the numerical solution of (1). From tables 1(a) and 1(b) we find that the difference between analytical and numerical calculations is around 10%.

Table 2. Some values for constants α_0^r in the case of SU(N) symmetry and $N=3,4$.

r	SU(3)	SU(4)
1	0.2206 356	0.2576 995
2	0.2206 356	0.3279 582
3		0.2576 995

Now let us consider the O($2N$) magnets. The nested BAE for integrable O($2N$) magnets are [9]:

$$\prod_{\substack{p=1\\ j\neq p}}^{n_r} \frac{\lambda_j^r-\lambda_p^r-\mathrm{i}}{\lambda_j^r-\lambda_p^r+\mathrm{i}} \prod_{l\in L_r}\prod_{p=1}^{n_l} \frac{\lambda_j^r-\lambda_p^l+\mathrm{i}[(m-1)\delta_{r,1}+1]/2}{\lambda_j^r-\lambda_p^l-\mathrm{i}[(m-1)\delta_{r,1}+1]/2}=1 \tag{10}$$

where $j=1,\ldots,n_r$; $r=1,\ldots,N-2,+,-$. Here $L_r=\{r-1,r+1\}$, $\{N-3,+,-\}$, $\{N-2\}$ for $1<r<N-3$, $r=N-2$, $r=+,-$ respectively; $\lambda_j^0=0$ and $n_0=L$. The densities $\sigma^r(x)$ for these O($2N$) magnets, in the limit $L\to\infty$ are [9, 10]:

$$\sigma^r(x)=\frac{2}{N-1}\left[\cos\left(\frac{\pi(N-r-1)}{2(N-1)}\right)\cosh\left(\frac{\pi x}{N-1}\right)\right]$$
$$\times\left[\cosh\left(\frac{2\pi x}{N-1}\right)+\cos\left(\frac{\pi(N-r-1)}{N-1}\right)\right]^{-1} \qquad r=1,\ldots,N-2 \tag{11}$$
$$\sigma^{(+,-)}(x)=\frac{1}{2(N-1)}\left[\cosh\left(\frac{\pi x}{(N-1)}\right)\right]^{-1}.$$

In this case the equations for the deviations are the same as (9) for $r<N-2$ and for $r=N-2,+,-$, they are

$$\begin{aligned}2\pi\alpha_p^{N-2}-\pi(\alpha_p^{N-3}+\alpha_p^{+}+\alpha_p^{-})&=-\log f_p^{N-2}\\ 2\pi\alpha_p^{+,-}-\pi(\alpha_p^{+,-}+\alpha_p^{N-2})&=-\log f_p^{+,-}.\end{aligned}\tag{12}$$

For $m=2$ the solutions of (9) ($r<N-2$) and (12) ($r=N-2,+,-$) are

$$\begin{aligned}&\alpha_0^r=r+1 \qquad r=1,\ldots,N-2\\ &\alpha_0^{+}=\alpha_0^{-}=\sqrt{N}.\end{aligned}\tag{13}$$

It is interesting to observe that for the O($2N$) group, α_0^1 is independent of N since this does not occur for the SU(N). In table 3 we show some values of α_0^r for O(6) and O(8) in the case $m=3$. In tables 4(a) and 4(b) we compare the numerical and analytical results for the deviations $\delta_0^r(x)$ for O(6) and O(8) with $m=2,3$ respectively. Again the differences are around 10%.

As a last remark it is convenient to rewrite (9) and (12) in a more simple form:

$$C\boldsymbol{\alpha}_p=-\log \boldsymbol{f}_p \tag{14}$$

where $\boldsymbol{\alpha}_p,\boldsymbol{f}_p$ are vectors with components $(\boldsymbol{\alpha}_p)_r=\alpha_p^r$, $(\boldsymbol{f}_p)_r=f_p^r$, and C is the Cartan matrix associated with Lie algebra O($2N$) and SU(N). A natural conjecture is that (14) continues to be valid for other simple Lie groups with an associated Cartan matrix C. The generalisation of this work for general Lie algebras and also the consideration of excited states we leave for a future work.

Table 3. Some values for constants α_0^r in the case of O($2N$) symmetry and $N=3,4$, with $\alpha_0^+=\alpha_0^-$.

r	O(6)	O(8)
1	0.3279 585	0.3366 371
+	0.2576 995	0.3366 371
2		0.5374 730

Table 4. The deviations for numerical (δ_0^r) and analytical (Δ_0^r) calculations for O($2N$) group with $N = 3, 4$ and (a) $m = 2$, (b) $m = 3$. Here we consider the lattice size $L = 24$ and the deviations are in the crescent order in index r ($\delta_0^1, \delta_0^2, \ldots, \delta_0^+$). We show only the results for δ_0^+, since $\delta_0^+ = \delta_0^-$ in the ground state.

(a)		$m = 2$		(b)		$m = 3$	
O(6)		O(8)		O(6)		O(8)	
δ_0^r	Δ_0^r	δ_0^r	Δ_0^r	δ_0^r	Δ_0^r	δ_0^r	Δ_0^r
0.116 44	0.113 46	0.178 95	0.171 00	0.173 53	0.175 80	0.272 42	0.268 84
0.046 78	0.052 03	0.066 49	0.069 27	0.070 08	0.080 88	0.101 86	0.108 61
0.027 35	0.028 62	0.037 43	0.038 29	0.040 84	0.043 61	0.057 25	0.059 37
0.019 07	0.019 68	0.024 72	0.025 15	0.028 41	0.029 78	0.037 80	0.038 81
0.014 49	0.014 84	0.017 88	0.018 12	0.021 58	0.022 38	0.027 31	0.027 90
0.011 56	0.011 89	0.013 78	0.013 93	0.017 34	0.017 88	0.021 03	0.021 42
0.009 78	0.009 95	0.011 71	0.011 28	0.014 55	0.014 95	0.017 05	0.017 34
0.008 51	0.008 64	0.009 46	0.009 55	0.012 65	0.012 96	0.014 35	0.014 66
0.007 63	0.007 74	0.008 32	0.008 40	0.011 35	0.011 61	0.012 70	0.012 89
0.007 05	0.007 14	0.007 58	0.007 65	0.010 48	0.010 71	0.015 67	0.011 73
0.006 69	0.006 78	0.007 13	0.007 19	0.009 95	0.010 16	0.010 88	0.011 03
0.006 52	0.006 60	0.006 92	0.006 98	0.009 69	0.009 90	0.010 56	0.010 71
0.107 25	0.118 95	0.206 73	0.176 18	0.159 13	0.173 05	0.315 76	0.277 21
0.038 24	0.039 73	0.110 28	0.124 97	0.056 62	0.060 07	0.169 85	0.198 91
0.023 99	0.024 56	0.066 02	0.067 14	0.035 42	0.036 76	0.101 88	0.105 45
0.018 38	0.018 69	0.047 78	0.049 35	0.027 11	0.027 86	0.073 65	0.077 29
0.015 78	0.015 99	0.037 43	0.038 25	0.023 26	0.023 79	0.057 63	0.059 64
0.014 70	0.014 87	0.030 97	0.031 54	0.021 66	0.022 11	0.047 66	0.049 05
		0.026 66	0.027 07			0.041 00	0.042 02
		0.023 70	0.024 01			0.036 44	0.037 23
		0.021 64	0.021 90			0.033 27	0.033 92
		0.020 52	0.020 47			0.031 13	0.031 70
		0.019 39	0.019 59			0.029 81	0.030 32
		0.018 98	0.019 17			0.029 17	0.029 66
		0.018 13	0.017 28			0.027 62	0.027 15
		0.071 68	0.074 44			0.109 79	0.116 53
		0.045 28	0.046 16			0.069 21	0.071 47
		0.034 76	0.035 22			0.053 08	0.054 28
		0.029 85	0.030 15			0.045 56	0.046 38
		0.027 82	0.028 06			0.042 44	0.043 12

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